:— State and Prove Logarithmic Test.

Theorem :— Suppose that $u_n > 0$ and that $\lim \left(n \log \frac{u_n}{u_{n+1}}\right) = K$

Then the Series is Convergent $K > 1$ and divergent if $K < 1$

Proof :— Let us Compare the given Series Σu_n with the auxiliary Series

$$\Sigma v_n = \frac{1}{1^P} + \frac{1}{2^P} + \frac{1}{3^P} + \cdots + \frac{1}{n^P} + \cdots$$

whose nth term $v_n = \frac{1}{n^P}$. We Know that the Series Σv_n is Convergent if $P > 1$ and is divergent if $P \leq 1$

$$\text{Now, } \frac{v_n}{v_{n+1}} = \frac{1}{n^P} \Big/ \frac{1}{(n+1)^P}$$

$$= \frac{(n+1)^P}{n^P} = \left(\frac{n+1}{n}\right)^P = \left(1+\frac{1}{n}\right)^P$$

I :— Suppose that Σv_n is Convergent and hence $P > 1$ then by the Comparison test Σu_n is Convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

i.e if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^P$

i.e if $\log \frac{u_n}{u_{n+1}} > P \log\left(1+\frac{1}{n}\right)$

i.e if $\log \frac{u_n}{u_{n+1}} > P\left(\frac{1}{n} - \frac{1}{2n^2} + \cdots\right)$

i.e if $n \log \frac{u_n}{u_{n+1}} > P - \frac{P}{2n} + \cdots$

i.e if $\lim \left(n \log \frac{u_n}{u_{n+1}}\right) \geq P (>1)$

i.e if $\lim \left(n \log \frac{u_n}{u_{n+1}}\right) > 1$

Suppose that Σv_n is divergent and hence $P < 1$

then Σu_n is divergent if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$

Evaluate :- $\int_0 \frac{\log(1+\cos a \cos x)}{\cos x} \cdot dx$

of :- let $I = \int_0^{\pi/2} \frac{\log(1+\cos a \cdot \cos x)}{\cos x} \cdot dx$ — (1)

Diff. w.r to a, we get

$$\frac{dI}{da} = \int_0^{\pi/2} \frac{(1-\sin a \cdot \cos x)}{(1+\cos a \cdot \cos x)} \cdot dx$$

$$= \int_0^{\pi/2} \frac{-\sin a}{(1+\cos a \cdot \cos x)} \cdot dx$$

But $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left[\sqrt{\frac{a-b}{a+b}} \tan x/2\right]_{a>b}$

therefore $\frac{dI}{da} = -\sin a \left[\frac{2}{\sqrt{1-\cos^2 a}} \tan^{-1}\left[\sqrt{\frac{1-\cos a}{1+\cos a}} \tan x/2\right]_0^{\pi/2}\right.$

$$= -2\tan^{-1}\left(\tan a/2\right) = -a$$

$$\Rightarrow dI = -a\,da$$

integrating $I = -\int a \cdot da = \frac{a^2}{2} + A$ — (2)

Putting $a = \frac{\pi}{2}$ in (1) we get $I = 0$

From (2) $0 - \frac{\pi^2}{8} + A$

$$\Rightarrow A = \frac{\pi^2}{8}$$

Therefore we get $I = \frac{1}{2}\left(\frac{\pi^2}{4} - a^2\right)$